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Slender, Two-Dimensional Bodies Having Minimum Total Drag at Hypersonic Speeds

Angelo Miele

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ANGELO MIELE AND ROBERT E. PRITCHARD

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SLENDER, TWO-DIMENSIONAL BODIES HAVING MINIMUM TOTAL DRAG AT HYPERSONIC SPEEDS

by

ANGELO MIELE[*] and ROBERT E. PRITCHARD[**]

SUMMARY

This paper considers the problem of minimizing the total drag (sum of the pressure drag and the friction drag) of a slender, two-dimensional, symmetric body at zero angle of attack in hypersonic flow under the assumption that the distribution of pressure coefficients is Newtonian and that the friction coefficient is constant. After the condition that the pressure coefficient must be nonnegative is accounted for, the minimal problem is solved for arbitrary conditions imposed on the thickness, the enclosed area, and the moment of inertia of the contour under the assumption that the length is free. It is shown that, if convenient dimensionless coordinates are employed (that is, if the abscissa and the ordinate are normalized with respect to the length and the semithickness), the totality of extremal arcs is composed of a two-parameter family of solutions. It is also shown that each extremal arc involves at most one corner point and, hence, two subarcs: one of these is characterized by a positive pressure coefficient and is called the regular shape; the other is characterized by a zero pressure coefficient and is called the zero-slope shape. Thus,

(*) Director of Astrodynamics and Flight Mechanics, Boeing Scientific Research Laboratories.

(**) Staff Associate, Boeing Scientific Research Laboratories.

two classes of bodies can be identified: (I) bodies composed of a regular shape only, and (II) bodies composed of a regular shape followed by a constant thickness contour.

Particular attention is devoted to solutions for which either one or two of the quantities under consideration are prescribed. If only one quantity is given (the thickness, the enclosed area, or the moment of inertia of the contour) the extremal arc is a single curve of class I regardless of the friction coefficient. On the other hand, if two geometric quantities are given (the thickness and the enclosed area, the thickness and the moment of inertia of the contour, or the enclosed area and the moment of inertia of the contour), a one-parameter family of extremal arcs exists; the parameter, called the friction parameter, is proportional to the cubic root of the friction coefficient and is related to the quantities which are prescribed. Depending on the value of the friction parameter, two distinct behaviors are possible. If the friction parameter is subcritical (smaller than a certain critical value), the solution is of class I; if the friction parameter is supercritical (larger than a certain critical value), the solution is of class II with the transition point from the regular shape to the constant thickness contour shifting forward as the friction parameter increases. For all of the cases considered, analytical expressions are derived for the optimum shapes, the thickness ratios, and the drag coefficients.

1. INTRODUCTION

The problem of minimizing the drag of slender, two-dimensional bodies in hypersonic flow has attracted considerable attention in recent times. With particular regard to the pressure drag, general solutions have been obtained in Ref. 1 under the assumption that the pressure distribution is Newtonian and that, among the geometrical quantities being considered (the thickness, the length, the enclosed area, and the moment of inertia of the contour), two are prescribed and the remaining two are free. These solutions have been extended in Ref. 2 to cover the case where three of these quantities are given and only one is free.

While the investigations of Refs. 1 and 2 neglected the friction drag, it should be noted that there exist practical values of the thickness ratio for which the friction drag may have the same order of magnitude as the pressure drag. Therefore, it is of interest to reinvestigate the problem of the optimum slender shape from the point of view of minimizing the total forebody drag (sum of the pressure drag and the friction drag) for any number of conditions imposed on the thickness d, the length ℓ, the enclosed area A, and the moment of inertia of the contour M. This is the problem considered in the present report in connection with the following assumptions: (a) the body is slender; (b) the distribution of pressure coefficients is Newtonian; and (c) the friction coefficient is constant along the contour. The corresponding axisymmetric problem is analyzed in Ref. 3 for any number of conditions imposed on the diameter, the length, the wetted area, and the volume.

2. MINIMUM DRAG PROBLEM

Consider a two-dimensional, symmetric airfoil at zero angle of attack in a hypersonic flow, and denote by x a coordinate in the flow direction, y a normal coordinate, and $\dot{y}$ the derivative dy/dx. Under the slender body approximation $\dot{y}^2 \ll 1$, the assumed Newtonian distribution of pressure coefficients simplifies to $C_p = 2\dot{y}^2$. Consequently, the drag per unit span of that portion of the body which is included between stations 0 and x is given by

$$D(x) = 4q\int_0^x \left(\dot{y}^3 + \frac{C_f}{2}\right) dx \tag{1}$$

where C_f is the friction coefficient, assumed constant. The corresponding values for the area enclosed by the contour and the moment of inertia of the contour are given by

$$A(x) = 2\int_0^x y dx \ , \quad M(x) = 2\int_0^x y^2 dx \tag{2}$$

After the definitions

$$\alpha = \frac{D(x)}{4q} \ , \quad \beta = \frac{A(x)}{2} \ , \quad \gamma = \frac{M(x)}{2} \tag{3}$$

are introduced, differentiation of both sides of Eqs. (1) and (2) with respect to the independent variable leads to the following differential constraints:

$$\dot{\alpha} - \dot{y}^3 - \frac{C_f}{2} = 0$$

$$\dot{\beta} - y = 0 \qquad (4)$$

$$\dot{\gamma} - y^2 = 0$$

Since the requirement that the slope be nonnegative everywhere can be expressed as

$$\dot{y} - p^2 = 0 \qquad (5)$$

where p denotes a real variable, the differential system composed of Eqs. (4) and (5) involves one independent variable (x), five dependent variables (y, α, β, γ, p), and one degree of freedom. In this connection, after assuming that

$$x_i = y_i = \alpha_i = \beta_i = \gamma_i = 0 \qquad (6)$$

and that some, but not all, of the remaining state variables are given at the final point, one can formulate the minimum drag problem as follows: <u>In the class of functions y(x), α(x), β(x), γ(x), p(x) which are consistent with the differential constraints (4) and (5) and the initial conditions (6), find that special set which minimizes the difference $\Delta G = G_f - G_i$, where $G = \alpha$.</u>

3. NECESSARY CONDITIONS

The previous problem is of the Mayer type with separated end conditions. Consequently, after the Lagrange multipliers λ_1 through λ_4 are introduced and the fundamental function is written as (Refs. 5 and 6)

$$F = \lambda_1\left(\dot{\alpha} - \dot{y}^3 - \frac{C_f}{2}\right) + \lambda_2(\dot{\beta} - y) + \lambda_3(\dot{\gamma} - y^2) + \lambda_4(\dot{y} - p^2) \tag{7}$$

the extremal arc is described by the following Euler-Lagrange equations:

$$\begin{aligned} \frac{d}{dx}(\lambda_4 - 3\lambda_1\dot{y}^2) &= -\lambda_2 - 2\lambda_3 y \\ \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= 0 \\ \dot{\lambda}_3 &= 0 \\ 0 &= \lambda_4 p \end{aligned} \tag{8}$$

the second, third, and fourth of which can be integrated to give

$$\lambda_1 = C_1 , \quad \lambda_2 = C_2 , \quad \lambda_3 = C_3 \tag{9}$$

where C_1 through C_3 are constants. Furthermore, after it is observed that the fundamental function does not contain the independent variable explicitly and after Eq. (8-5) is accounted for, the following first integral can be established:

$$C_1\left(\frac{C_f}{2} - 2\dot{y}^3\right) + C_2 y + C_3 y^2 = C \tag{10}$$

where C is a constant.

<u>Corner conditions</u>. As the fifth Euler equation indicates, the extremal arc is composed of the subarcs

$$\lambda_4 = 0 \quad \text{and/or} \quad p = 0 \tag{11}$$

Along the former subarcs, called <u>regular shapes</u>, the pressure coefficient is always positive as long as p is real. Along the latter subarcs, called <u>zero-slope shapes</u>, the pressure coefficient is always zero. The junctions between the subarcs must be studied with the aid of the Erdmann-Weierstrass corner conditions. They require that the integration constants C_1, C_2, C_3, C have the same value for each of the subarcs composing the extremal arc and that(*)

$$\Delta(\dot{y}^3) = \Delta(\lambda_4 - 3C_1\dot{y}^2) = 0 \tag{12}$$

where $\Delta(\ldots)$ denotes the difference between quantities evaluated after and before the corner point. A mathematical consequence of these equations are the relationships

$$\Delta\lambda_4 = \Delta\dot{y} = 0 \tag{13}$$

(*)Eq. (12-1) is a consequence of the first integral (10) and the continuity of the integration constants.

which, if combined with Eqs. (11), imply that

$$\lambda_4 = \dot{y} = 0 \tag{14}$$

before and after a corner point.

End conditions. The end conditions are partly of the fixed end-point type and partly of the natural type. The latter must be determined from the transversality condition

$$\Big[- C dx + (C_1 + 1)\, d\alpha + C_2 d\beta + C_3 d\gamma + (\lambda_4 - 3C_1\dot{y}^2)\, dy\Big]_i^f = 0 \tag{15}$$

which must be satisfied for every system of differentials consistent with the prescribed end conditions; in particular, it implies that $C_1 = -1$.

If the length is free, the transversality condition yields $C = 0$. On the other hand, if either the enclosed area or the moment of inertia of the contour are free, the transversality condition leads to $C_2 = 0$ and $C_3 = 0$, respectively. Finally, if the diameter is free, the transversality condition leads to

$$(\lambda_4 + 3\dot{y}^2)_f = 0 \tag{16}$$

which, if combined with the Euler-Lagrange equation (8-5), implies that

$$\lambda_{4f} = \dot{y}_f = 0 \tag{17}$$

Consequently, if $d = 2y_f$ denotes the diameter, the first integral (10) yields the additional relationship

$$C + \frac{C_f}{2} - C_2 \frac{d}{2} - C_3 \frac{d^2}{4} = 0 \tag{18}$$

At this point, it is convenient to separate the discussion into two basic problems: problems where the length is given and problems where the length is free. As Eq. (1) shows, problems of the first kind are characterized by the fact that the friction drag is independent of the shape so that the contour which minimizes the total drag is identical with that which minimizes the pressure drag. Since shapes of minimum pressure drag have been fully discussed in Refs. 1 and 2, these problems are not considered here. Thus, the analysis is restricted to problems of the second kind, in which the length is free. The class of problems in which the length is free contains several subclasses which depend on the number of quantities that are specified. Among these subclasses, the following are considered here: problems in which one geometric quantity is given and problems in which two geometric quantities are given. For these problems, simple manipulations lead to the results which are summarized in Table 1 where two types of relations are indicated: those obtained from the transversality condition and those obtained by combining the results of the transversality condition with the Euler-Lagrange equation (8-5) and the first integral (10).

Legendre-Clebsch condition. The Legendre-Clebsch condition indicates that the drag is a minimum if the following inequalities are satisfied everywhere along the extremal arc:

$$\dot{y} \geq 0 \,, \quad \text{along the regular shape}$$
$$\lambda_4 \leq 0 \,, \quad \text{along the zero-slope shape} \tag{19}$$

<u>Switching function</u>. From the previous discussion, it appears that the Lagrange multiplier λ_4 plays an important role in determining the composition of the extremal arc. If the terminology of control theory is employed, this multiplier can be called the switching function; its properties are as follows:

$$\lambda_4 = 0 \,, \quad \text{along the regular shape}$$
$$\lambda_4 \leq 0 \,, \quad \text{along the zero-slope shape} \tag{20}$$
$$\lambda_4 = 0 \,, \quad \text{at a corner point}$$

4. GEOMETRY OF THE EXTREMAL ARC

In the previous section, the necessary conditions to be satisfied by the extremal arc have been stated. In this section, several general consequences of these equations are derived referring, for the sake of brevity, to the minimum drag problem ($C_1 = -1$) with the length unspecified ($C = 0$). In order to facilitate the analysis, the following dimensionless coordinates are introduced:

$$\xi = x/\ell \ , \quad \eta = 2y/d \tag{21}$$

together with the definitions

$$K_2 = C_2 d/C_f \ , \quad K_3 = C_3 d^2/2C_f \tag{22}$$

With these coordinates, the first integral (10) reduces to the form

$$\frac{\tau^3}{2C_f}\dot{\eta}^3 = 1 - K_2\eta - K_3\eta^2 \tag{23}$$

where $\tau = d/\ell$ denotes the thickness ratio and $\dot{\eta}$ the derivative $d\eta/d\xi$.

<u>Basic inequality</u>. The application of the above first integral at the end points of the extremal arc indicates that the terminal values of the slope are given by

$$\dot{\eta}_i = \frac{\sqrt[3]{2C_f}}{\tau}$$
$$\dot{\eta}_f = \frac{\sqrt[3]{2C_f}}{\tau}(1 - K_2 - K_3)^{1/3} \tag{24}$$

From the first equation, it is apparent that the optimum body is sharp-nosed. Since the final slope (and, hence, the final pressure coefficient) must be nonnegative, one deduces from the second equation that the following basic inequality must be satisfied:

$$1 - K_2 - K_3 \geq 0 \tag{25}$$

Switching function. Since each extremal arc may involve more than one subarc, it is of paramount importance to calculate the distribution of the switching function; in nondimensional form, this function can be defined as

$$\sigma = \frac{d}{\ell C_f} \lambda_4 \tag{26}$$

For the regular shape, it is known that $\sigma = 0$. For the zero-slope shape, it is known that $\dot{\eta} = 0$ and $\eta = \text{const}$. Consequently, the Euler-Lagrange equation (8-1) reduces to

$$\dot{\sigma} = -K_2 - 2K_3\eta \tag{27}$$

which, in the light of the initial conditions (20-3), admits the particular integral

$$\sigma = -(K_2 + 2K_3\eta_c)(\xi - \xi_c) \tag{28}$$

in which the subscript c refers to a corner point.

Sequence of subarcs. The next step is to determine the appropriate sequence of subarcs. First, it is observed that the extremal arc cannot start with the zero-slope shape $\eta = 0$: the equation of this shape would be incompatible with the first integral (23). Thus, if it is assumed that the extremal arc starts with a regular shape, the next questions to be investigated are: (a) whether a transition to a zero-slope shape is possible; and (b) if so, whether a further transition back to a regular shape may occur. Concerning the first question, the corner condition (14-2) and the first integral (23) show that the transition from the regular shape to the zero-slope shape is possible if the following relationship is satisfied:

$$1 - K_2\eta_c - K_3\eta_c^2 = 0 \tag{29}$$

With regard to the second question, Eq. (28) shows that, since the switching function varies linearly with the abscissa along the zero-slope shape, it can only vanish at one point: the corner point between the regular shape and the zero-slope shape. Thus, no regular shape may follow the zero-slope shape so that the equation of the latter is $\eta = 1$. Furthermore, because of Eq. (29) and the properties of the switching function, the presence of a zero-slope shape requires that

$$\begin{gathered} 1 - K_2 - K_3 = 0 \\ K_2 + 2K_3 \geq 0 \end{gathered} \tag{30}$$

Since no more than one corner point and two subarcs may exist, the totality of extremal arcs consists of two classes of bodies: (I) bodies composed of a regular shape only, and (II) bodies composed of a regular shape followed by a constant thickness contour. These bodies are represented symbolically by

$$\begin{aligned} &\text{Class I}: \quad \sigma = 0 \\ &\text{Class II}: \quad \sigma = 0 \rightarrow \eta = 1 \end{aligned} \tag{31}$$

Family of solutions. Since the most general type of extremal arc is of class II, its geometry can be described by the equations

$$0 \le \xi \le \xi_c \,, \quad \frac{\xi}{\xi_c} = \frac{\int_0^{\eta} (1 - K_2\eta - K_3\eta^2)^{-1/3}\, d\eta}{\int_0^{1} (1 - K_2\eta - K_3\eta^2)^{-1/3}\, d\eta} \tag{32}$$

$$\xi_c \le \xi \le 1 \,, \quad \eta = 1$$

where ξ_c denotes the abscissa of the transition point. Bodies of class I can be obtained from bodies of class II by means of the formal substitution $\xi_c = 1$; it should be noted, however, that the corner condition need not be satisfied at this special point. In a functional form, Eqs. (32) can be rewritten as

$$\eta = \eta(\xi, \xi_c, K_2, K_3) \tag{33}$$

so that, after this relationship is combined with either of the relations

$$\begin{aligned} &\text{Class I :} \quad \xi_c = 1 \\ &\text{Class II :} \quad 1 - K_2 - K_3 = 0 \end{aligned} \tag{34}$$

it is seen that a two-parameter family of optimum bodies exists.

For particular types of boundary conditions, considerable simplifications are possible. Thus, if the enclosed area is free ($K_2 = 0$) or the moment of inertia of the contour is free ($K_3 = 0$), the number of independent parameters is reduced by one. An analogous remark holds for the case where the thickness is free, since $K_2 + K_3 = 1$. In conclusion, the number of independent parameters governing the solution depends on the number of geometric quantities other than the length which are prescribed. If three quantities are prescribed, the problem admits a two-parameter family of solutions. If two quantities are prescribed, the problem admits a one-parameter family of solutions. Finally, if only one quantity is prescribed, the problem admits a zero-parameter family of solutions, that is, the geometry of the extremal arc in the $\xi\eta$-plane consists of a single curve regardless of the value of the friction coefficient. In connection with problems where either one or two geometric quantities are prescribed, the set of dimensionless boundary conditions is indicated in Table 2 along with the dimensionless switching function at the final point, the slope of the extremal arc at the final point, and the number of independent parameters governing the solution.

5. SOLUTION OF THE BOUNDARY VALUE PROBLEM

In this section, a general method for determining the unknowns appearing in Eqs. (32) is presented. The analysis is facilitated if several nondimensional integrals are introduced. If the cubic root of both sides of Eq. (23) is extracted, if the variables are separated, and if an integration over the regular shape is performed, the following result is obtained:

$$\sqrt[3]{2C_f}/\tau = I_d(\xi_c, K_2, K_3) \tag{35}$$

where

$$\tau = d/\ell \tag{36}$$

denotes the thickness ratio and I_d the nondimensional integral

$$I_d(\xi_c, K_2, K_3) = \frac{1}{\xi_c}\int_0^1 (1 - K_2\eta - K_3\eta^2)^{-1/3}\, d\eta \tag{37}$$

Furthermore, by simple manipulations, the area enclosed by the airfoil and the moment of inertia of the contour become

$$\begin{aligned} A\tau/d^2 &= I_A(\xi_c, K_2, K_3) \\ 2M\tau/d^3 &= I_M(\xi_c, K_2, K_3) \end{aligned} \tag{38}$$

where

$$I_A(\xi_c, K_2, K_3) = \int_0^1 \eta d\xi = 1 - \xi_c \frac{\int_0^1 (1 - \eta)(1 - K_2\eta - K_3\eta^2)^{-1/3} d\eta}{\int_0^1 (1 - K_2\eta - K_3\eta^2)^{-1/3} d\eta}$$

$$\tag{39}$$

$$I_M(\xi_c, K_2, K_3) = \int_0^1 \eta^2 d\xi = 1 - \xi_c \frac{\int_0^1 (1 - \eta^2)(1 - K_2\eta - K_3\eta^2)^{-1/3} d\eta}{\int_0^1 (1 - K_2\eta - K_3\eta^2)^{-1/3} d\eta}$$

For a given friction coefficient, the system composed of the five equations (34) through (36) and (38) involves the eight quantities

$$\tau,\ d,\ \ell,\ A,\ M,\ \xi_c,\ K_2,\ K_3 \tag{40}$$

which means that one particular optimum body can be determined if three additional relationships are specified. For the boundary conditions considered in Table 2, these relationships are represented by any one of the following sets:

$$\begin{array}{lll} d - \text{Const} = 0\ , & K_2 = 0 & ,\ K_3 = 0 \\ A - \text{Const} = 0\ , & K_2 = 1 & ,\ K_3 = 0 \\ M - \text{Const} = 0\ , & K_2 = 0 & ,\ K_3 = 1 \\ d - \text{Const} = 0\ , & A - \text{Const} = 0\ , & K_3 = 0 \\ d - \text{Const} = 0\ , & M - \text{Const} = 0\ , & K_2 = 0 \\ A - \text{Const} = 0\ , & M - \text{Const} = 0\ , & K_2 = 1 - K_3 \end{array} \tag{41}$$

<u>Drag coefficient</u>. After the boundary value problem has been solved, the next step is to determine the drag of the optimum body. This drag can be written as

$$D = \frac{qd\tau^2}{2}\left(I_{Dp} + \frac{4C_f}{\tau^3}\right) \tag{42}$$

where I_{Dp} denotes the dimensionless integral

$$I_{Dp} = \int_0^1 \dot{\eta}^3 d\xi \tag{43}$$

Now, if the drag coefficient is referred to the frontal area at $x = \ell$ (that is, if $C_D = D/qd$), the following relationship can be readily established between the drag coefficient, the friction coefficient, and the thickness ratio:

$$\frac{C_D}{\tau^2} = \frac{I_{Dp}}{2} + 2\frac{C_f}{\tau^3} \tag{44}$$

Notice that, if both sides of Eq. (23) are multiplied by $d\xi$ and integrated over the entire length of the extremal arc, the relationship

$$\frac{\tau^3}{2C_f} I_{Dp} = 1 - K_2 I_A - K_3 I_M \tag{45}$$

can be obtained. Consequently, after Eqs. (35), (44), and (45) are combined, one deduces that

$$\frac{C_D}{\tau^2} = \frac{I_d^3}{2} (3 - K_2 I_A - K_3 I_M) \tag{46}$$

<u>Drag ratio</u>. Another interesting quantity characterizing the optimum body is the drag ratio, that is, the ratio of the friction drag to the total drag. Because of Eq. (44), this quantity is given by

$$\frac{C_{Df}}{C_D} = \frac{4C_f}{4C_f + \tau^3 I_{Dp}} \tag{47}$$

which, in the light of Eq. (45), can be rewritten as

$$\frac{C_{Df}}{C_D} = \frac{2}{3 - K_2 I_A - K_3 I_M} \tag{48}$$

6. PARTICULAR CASES

In the previous sections, the minimum drag problem was solved in general for arbitrary boundary conditions. Here, several particular cases are considered, and the associated optimum shapes are calculated. Two classes of problems are considered: (a) problems in which only one geometric quantity is prescribed and (b) problems in which two geometric quantities are prescribed. Problems of type (a) are characterized by solutions of class I, that is, solutions involving a regular shape only. Problems of type (b) are characterized by solutions of either class I (regular shape) or class II (regular shape followed by a constant thickness contour) depending on whether the friction coefficient is smaller or larger than a certain critical value.

6.1. Given Thickness

If the thickness is given while the enclosed area and the moment of inertia of the contour are free, the transversality condition leads to $K_2 = K_3 = 0$. Since Eq. (30-1) is not satisfied, a zero-slope shape cannot exist. Hence, the extremal arc is of class I, that is, involves a regular shape only.

After setting $\xi_c = 1$, the equation of the regular shape (32-1) can be integrated to give (Fig. 1)

$$\eta = \xi \tag{49}$$

meaning that the extremal arc is a wedge. Since $I_d = 1$, Eq. (35) yields the following value for the optimum thickness ratio:

$$\tau = \sqrt[3]{2C_f} = 1.26\,\sqrt[3]{C_f} \tag{50}$$

Consequently, Eq. (46) indicates that the drag coefficient per unit thickness ratio squared is given by

$$\frac{C_D}{\tau^2} = \frac{3}{2} = 1.5 \tag{51}$$

a result which, because of Eq. (48), has the following implication: the friction drag of the optimum body is two-thirds of the total drag.

6.2. Given Enclosed Area

If the enclosed area is given while the thickness and the moment of inertia of the contour are free, the transversality condition leads to $K_2 = 1$, $K_3 = 0$, and $\sigma_f = 0$. Should a zero-slope shape exist, the switching function would be zero at both ends of this subarc. However, because of Eq. (28), this is only possible when $\xi_f = \xi_c$. Since the length of the zero-slope shape is zero, the extremal arc is of class I, that is, includes a regular shape only.

For $\xi_c = 1$, the equation of the regular shape (32-1) can be integrated to give (Fig. 1)

$$\eta = 1 - (1 - \xi)^{3/2} \tag{52}$$

Since $I_d = 3/2$, Eq. (35) yields the following value for the optimum thickness ratio:

$$\tau = \frac{2}{3}\sqrt[3]{2C_f} = 0.84\sqrt[3]{C_f} \tag{53}$$

Finally, since $I_A = 3/5$, Eq. (46) yields the following minimum drag coefficient:

$$\frac{C_D}{\tau^2} = \frac{81}{20} = 4.05 \tag{54}$$

which, in the light of Eq. (48), has the following implication: the friction drag of the optimum body is five-sixths of the total drag.

6.3. Given Moment of Inertia of the Contour

If the moment of inertia of the contour is given while the thickness and the enclosed area are free, the transversality condition leads to $K_2 = 0$, $K_3 = 1$, and $\sigma_f = 0$. Should a zero-slope shape exist, the switching function would be zero at both ends of this subarc. However, because of Eq. (28), this is only possible when $\xi_f = \xi_c$. Since the length of the zero-slope shape is zero, the extremal arc is of class I, that is, includes a regular shape only.

For $\xi_c = 1$, one can integrate the equation of the regular shape (32-1) to obtain (Fig. 1)

$$\xi = \frac{f(\eta)}{f(1)} \tag{55}$$

where

$$f(\eta) = -\frac{\sqrt{3}-1}{2\sqrt[4]{3}}F(\varphi, k) + \sqrt[4]{3}E(\varphi, k) - \frac{\eta}{\sqrt{3}+1-\sqrt[3]{1-\eta^2}} \tag{56}$$

and where F and E denote the incomplete elliptic integrals of the first and second kind, respectively. The associated argument φ and parameter k are defined as

$$\varphi(\eta) = \arccos\frac{\sqrt{3}-1+\sqrt[3]{1-\eta^2}}{\sqrt{3}+1-\sqrt[3]{1-\eta^2}}, \quad k = \sqrt{\frac{2+\sqrt{3}}{4}} \tag{57}$$

Since $I_d = 3f(1)$, Eq. (35) yields the following value for the optimum thickness ratio:

$$\tau = \frac{\sqrt[3]{2C_f}}{3f(1)} = 0.97\sqrt[3]{C_f} \tag{58}$$

Finally, since $I_M = 3/7$, the minimum drag coefficient becomes

$$\frac{C_D}{\tau^2} = \frac{243}{7}f^3(1) = 2.78 \tag{59}$$

a result which, because of Eq. (48), has the following implication: the friction drag of the optimum body is seven-ninths of the total drag.

6.4. Given Thickness and Enclosed Area

If the thickness and the enclosed area are prescribed while the moment of inertia of the contour is free, the transversality condition leads to

$K_3 = 0$. The totality of extremal arcs is represented by a one-parameter family of solutions of either class I (regular shape) or class II (regular shape followed by a constant thickness contour) depending on whether the friction coefficient is smaller or larger than a certain critical value. The representation of the results becomes simple and immediate if a thickness parameter and a friction parameter are introduced. These parameters are defined by

$$K_\tau = \tau \frac{A}{d^2}, \quad K_f = \sqrt[3]{2C_f}\,\frac{A}{d^2} \tag{60}$$

and, because of Eqs. (35) and (38-1), can be rewritten as

$$K_\tau = I_A(\xi_c, K_2), \quad K_f = I_d(\xi_c, K_2)\, I_A(\xi_c, K_2) \tag{61}$$

<u>Bodies of class I.</u> These bodies consist of a regular shape only and are obtained for $\xi_c = 1$ and $-\infty \le K_2 \le 1$. After Eqs. (32-1), (46), and (61) are employed, the optimum shape, the thickness parameter, the drag coefficient, and the friction parameter can be rewritten as

$$\xi = \frac{1 - (1 - K_2\eta)^{2/3}}{1 - (1 - K_2)^{2/3}}$$

$$K_\tau = \frac{1}{5K_2}\,\frac{3 - (3 + 2K_2)(1 - K_2)^{2/3}}{1 - (1 - K_2)^{2/3}}$$

$$\frac{C_D}{\tau^2} = \frac{1}{5}\left(\frac{3}{2K_2}\right)^3 \left[1 - (1 - K_2)^{2/3}\right]^2 \left[6 - (6 - K_2)(1 - K_2)^{2/3}\right] \tag{62}$$

$$K_f = \frac{3}{10K_2^2}\left[3 - (3 + 2K_2)(1 - K_2)^{2/3}\right]$$

Elimination of the parameter K_2 from these equations yields the functional relationships

$$\eta = \eta(\xi, K_f) , \quad K_\tau = K_\tau(K_f) , \quad \frac{C_D}{\tau^2} = \frac{C_D}{\tau^2}(K_f) \tag{63}$$

which are represented in Figs. 2 through 4 and are valid in the interval $0 \le K_f \le 9/10$. Incidentally, the solution corresponding to $K_f = 1/2$ is a wedge.

<u>Bodies of class II</u>. These bodies consist of a regular shape followed by a constant thickness contour and are obtained for $0 \le \xi_c \le 1$ and $K_2 = 1$. The shape of the optimum body, the thickness parameter, the drag coefficient, and the friction parameter are written as

$$0 \le \xi \le \xi_c , \quad \eta = 1 - \left(1 - \frac{\xi}{\xi_c}\right)^{3/2}$$

$$\xi_c \le \xi \le 1 , \quad \eta = 1$$

$$K_\tau = 1 - \frac{2}{5}\xi_c \tag{64}$$

$$\frac{C_D}{\tau^2} = \left(\frac{3}{2\xi_c}\right)^3 \left(1 + \frac{\xi_c}{5}\right)$$

$$K_f = \frac{3}{10}\left(\frac{5}{\xi_c} - 2\right)$$

Elimination of the abscissa of the transition point from these equations leads once more to functional relationships of the form (63) which are plotted in Figs. 2 through 4 and are valid for $9/10 \le K_f \le \infty$. As the friction parameter increases, the abscissa of the transition point moves forward, a circumstance which is common to all of the boundary value problems analyzed here.

6.5. Given Thickness and Moment of Inertia of the Contour

If the thickness and the moment of inertia of the contour are prescribed while the enclosed area is free, the transversality condition leads to $K_2 = 0$. As in the previous case, a one-parameter family of solutions exists. These solutions are of either class I or class II depending on whether the friction coefficient is smaller or larger than a certain critical value. Once more, the representation of the results is facilitated by introducing a thickness parameter and a friction parameter. These parameters are defined as

$$K_\tau = \frac{2M}{d^3}\tau , \quad K_f = \frac{2M}{d^3}\sqrt[3]{2C_f} \tag{65}$$

and, because of Eqs. (35) and (38-2), can be rewritten as

$$K_\tau = I_M(\xi_c, K_3) , \quad K_f = I_d(\xi_c, K_3)\, I_M(\xi_c, K_3) \tag{66}$$

Bodies of Class I. These bodies consist of a regular shape only and are obtained for $\xi_c = 1$ and $-\infty \le K_3 \le 1$. Using Eq. (32-1), one can express the geometry of the optimum shape in the form

$$\xi = \frac{g(\eta, K_3)}{g(1, K_3)} \tag{67}$$

where

$$g(\eta, K_3) = \pm\left[\frac{\sqrt{\mp K_3}\,\eta}{\sqrt{3} \mp 1 \pm \sqrt[3]{1 - K_3\eta^2}} + \frac{\sqrt{3} \pm 1}{2\sqrt[4]{3}} F(\varphi, k) - \sqrt[4]{3}\, E(\varphi, k)\right] \tag{68}$$

and where the upper sign is valid for $K_3 \leq 0$ and the lower sign, for $K_3 \geq 0$. The symbols F and E denote the incomplete elliptic integrals of the first and second kind whose arguments φ and parameter k are defined as

$$\varphi(\eta, K_3) = \arc\cos \frac{\sqrt{3} \pm 1 \mp \sqrt[3]{1 - K_3\eta^2}}{\sqrt{3} \mp 1 \pm \sqrt[3]{1 - K_3\eta^2}}, \quad k = \sqrt{\frac{2 \mp \sqrt{3}}{4}} \tag{69}$$

Because of Eqs. (46) and (65), the thickness parameter, the drag coefficient, and the friction parameter can be expressed in the form

$$K_\tau = \frac{3}{7K_3}\left[1 - \frac{\sqrt{\mp K_3}\,(1 - K_3)^{2/3}}{3g(1, K_3)}\right]$$

$$\frac{C_D}{\tau^2} = \frac{27}{14}\,\frac{g^2(1, K_3)}{(\mp K_3)^{3/2}}\left[18g(1, K_3) + \sqrt{\mp K_3}\,(1 - K_3)^{2/3}\right] \tag{70}$$

$$K_f = \frac{3}{7K_3\sqrt{\mp K_3}}\left[3g(1, K_3) - \sqrt{\mp K_3}\,(1 - K_3)^{2/3}\right]$$

Elimination of the parameter K_3 from Eqs. (67) and (70) yields functional relationships of the form (63) which are plotted in Figs. 5 through 7 and are valid in the interval $0 \le K_f \le 0.55$. Incidentally, the solution corresponding to $K_f = 1/3$ is a wedge.

Bodies of class II. These bodies consist of a regular shape followed by a constant thickness contour and are obtained for $0 \le \xi_c \le 1$ and $K_3 = 1$. After the shape of the optimum body, the thickness parameter, the drag coefficient, and the friction parameter are written as

$$0 \le \xi \le \xi_c\,, \qquad \frac{\xi}{\xi_c} = \frac{g(\eta, 1)}{g(1, 1)}$$

$$\xi_c \le \xi \le 1\,, \qquad \eta = 1$$

$$K_\tau = 1 - \frac{4}{7}\xi_c \tag{71}$$

$$\frac{C_D}{\tau^2} = \left[\frac{3g(1, 1)}{\xi_c}\right]^3 \left(1 + \frac{2}{7}\xi_c\right)$$

$$K_f = \frac{3g(1, 1)}{\xi_c}\left(1 - \frac{4}{7}\xi_c\right)$$

elimination of the abscissa of the transition point from these equations leads once more to functional relationships of the form (63) which are plotted in Figs. 5 through 7 and are valid in the interval $0.55 \le K_f \le \infty$.

6.6. Given Enclosed Area and Moment of Inertia of the Contour

If the enclosed area and the moment of inertia of the contour are given while the thickness is free, the transversality condition leads to

$K_2 = 1 - K_3$. Again, the extremal arc is represented by a one-parameter family of solutions of either class I or class II depending on whether the friction coefficient is smaller or larger than a certain critical value. Once more, the representation of the results is facilitated if a thickness parameter and a friction parameter are introduced. These parameters are defined by

$$K_\tau = \frac{A^3}{4M^2}\tau\,, \quad K_f = \frac{A^3}{4M^2}\sqrt[3]{2C_f} \tag{72}$$

and, because of Eqs. (38), can be rewritten as

$$K_\tau = \frac{I_A^3(\xi_c, K_3)}{I_M^2(\xi_c, K_3)}\,, \quad K_f = \frac{I_d(\xi_c, K_3)\, I_A^3(\xi_c, K_3)}{I_M^2(\xi_c, K_3)} \tag{73}$$

<u>Bodies of class I</u>. These bodies consist of a regular shape only and are obtained for $\xi_c = 1$ and $\infty \geq K_3 \geq -1$. If Eq. (32-1) is employed, the expression for the optimum shape can be written as

$$\xi = \frac{h(0, K_3) \mp h(\eta, K_3)}{h(0, K_3) \mp h(1, K_3)} \tag{74}$$

In the numerator of the above equation, the upper sign is valid for $\eta \leq (K_3 - 1)/2K_3$, and the lower sign, for $\eta \geq (K_3 - 1)/2K_3$ if $\infty \geq K_3 \geq 1$; for $1 \geq K_3 \geq -1$, the upper sign is valid for all values of η. With regard to the denominator, the lower sign is valid when $\infty \geq K_3 \geq 1$, and the upper sign, when $1 \geq K_3 \geq -1$. The definition of the function $h(\eta, K_3)$ which appears in Eq. (74) is given by

$$h(\eta, K_3) = \mp \left[\frac{\sqrt{1 - \alpha^3}}{\sqrt{3} + 1 - \alpha} + \frac{\sqrt{3} - 1}{2\sqrt[4]{3}} F(\varphi, k) - \sqrt[4]{3}\, E(\varphi, k) \right] \tag{75}$$

where the upper sign is valid for $\infty \geq K_3 \geq 1$, the lower sign is valid for $1 \geq K_3 \geq -1$, and the quantities α, φ, and k are defined as

$$\alpha(\eta, K_3) = \left[\frac{4K_3(1 + K_3\eta)(1 - \eta)}{(1 + K_3)^2} \right]^{1/3}$$

$$\varphi(\eta, K_3) = \text{arc cos} \frac{\sqrt{3} - 1 + \alpha}{\sqrt{3} + 1 - \alpha}, \qquad k = \sqrt{\frac{2 + \sqrt{3}}{4}} \tag{76}$$

Finally, because of Eqs. (46) and (73), the thickness parameter, the drag coefficient, and the friction parameter can be expressed in the form

$$K_\tau = \frac{\gamma^3}{\beta\delta^2}$$

$$\frac{C_D}{\tau^2} = \frac{\beta^3}{2} \left[3 - (1 - K_3) \frac{\gamma}{\beta} - K_3 \frac{\delta}{\beta} \right] \tag{77}$$

$$K_f = \frac{\gamma^3}{\delta^2}$$

where

$$\beta(K_3) = 3\sqrt[s]{\frac{1+K_3}{2K_3^2}}\left[h(0, K_3) \mp h(1, K_3)\right]$$

$$\gamma(K_3) = \frac{3}{2K_3}\sqrt[s]{\frac{1+K_3}{2K_3^2}}\left\{-(1-K_3)\left[h(0, K_3) \mp h(1, K_3)\right]\right\} + \frac{3}{4K_3} \tag{78}$$

$$\delta(K_3) = \frac{3}{7K_3}\left[\beta - \frac{5}{3}(1-K_3)\,\gamma\right]$$

and where the lower signs are valid for $\infty \geq K_3 \geq 1$ and the upper signs, for $1 \geq K_3 \geq -1$. Elimination of the parameter K_3 from Eqs. (74) and (77) yields functional relationships of the form (63) which are plotted in Figs. 8 through 10 and are valid for $0 \leq K_f \leq 49/16$.

<u>Bodies of class II</u>. These bodies consist of a regular shape followed by a constant thickness contour and are obtained for $0 \leq \xi_c \leq 1$, $K_2 = 2$, and $K_3 = -1$. After the shape of the optimum body, the thickness parameter, the drag coefficient, and the friction parameter are written as

$$0 \leq \xi \leq \xi_c\ , \quad \eta = 1 - \left(1 - \frac{\xi}{\xi_c}\right)^3$$

$$\xi_c \leq \xi \leq 1\ , \quad \eta = 1$$

$$K_\tau = \frac{49}{16}\frac{(4-\xi_c)^3}{(14-5\xi_c)^2}$$

$$\frac{C_D}{\tau^2} = \left(\frac{3}{\xi_c}\right)^3\left(1 + \frac{\xi_c}{14}\right) \tag{79}$$

$$K_f = \frac{147}{16\xi_c}\frac{(4-\xi_c)^3}{(14-5\xi_c)^2}$$

elimination of the abscissa of the transition point from these equations leads once more to functional relationships of the form (63) which are plotted in Figs. 8 through 10 and are valid for $49/16 \leq K_f \leq \infty$.

7. DISCUSSION AND CONCLUSIONS

From the previous analysis, it appears that, despite the generality of the present problem, the method of solution is relatively simple and has the merit of leading to analytical solutions in each of the six particular cases considered here. The main comments to these solutions are as follows:

(a) For the general problem in which the length is free and arbitrary conditions are assigned to the thickness, the enclosed area, and the moment of inertia of the contour, the totality of extremal arcs is represented by a two-parameter family of solutions if dimensionless coordinates are employed, that is, if the abscissa and the ordinate are normalized with respect to the length and the semithickness. Each member of the family is characterized by a sharp leading edge. Furthermore, each extremal arc may involve at most one corner point and, hence, two subarcs. Of these subarcs, one is characterized by a positive pressure coefficient and is called the regular shape; the other is characterized by a zero pressure coefficient and is called the zero-slope shape. Consequently, two classes of bodies can be identified: (I) bodies composed of a regular shape only and (II) bodies composed of a regular shape followed by a constant thickness contour.

(b) If only one geometric quantity is assigned (the thickness, the enclosed area, or the moment of inertia of the contour), a zero-parameter family of solutions exists (that is, a single curve). In all cases, the solution is of class I, that is, consists of a regular shape only. In particular, if the thickness is given, the solution is a wedge, and its length is such that the friction drag is 2/3 of the total drag. If the enclosed area is given, the complements of the ordinate and the abscissa obey a 3/2 power law, and the optimum thickness ratio is such that the friction drag is 5/6 of the total drag. Finally, if the moment of inertia of the contour is given, the solution is represented by a combination of elliptic integrals of the first and the second kind, and the optimum thickness ratio is such that the friction drag is 7/9 of the total drag.

(c) If two geometric quantities are prescribed (the thickness and the enclosed area, the thickness and the moment of inertia of the contour, or the enclosed area and the moment of inertia of the contour), a one-parameter family of solutions exists. This parameter, called the friction parameter, is proportional to the cubic root of the friction coefficient and is indicative of the relative importance of the friction drag with respect to the pressure drag. Depending on the value of the friction parameter, two distinct behaviors are possible. If the friction parameter is subcritical (smaller than a certain critical value), the solution is of class I and, therefore, involves a regular shape only. If the friction parameter is supercritical (larger than a certain critical value), the solution is of class II and, therefore, involves a regular shape followed by a constant thickness contour; in all cases, the transition point from the regular shape to the constant thickness contour shifts forward as the friction parameter increases.

In closing, it should be noted that, if the limiting process $C_f \rightarrow 0$ is carried out, the present solutions reduce to the inviscid flow solutions already calculated in Ref. 1. It should also be noted that some of the optimum shapes obtained with this analysis are concave; consequently, these bodies should be restudied using the Newton-Busemann pressure coefficient law; this, however, requires a more thorough understanding of the friction drag associated with the possible presence of free layers. Finally, when the square of the thickness ratio becomes nonnegligible with respect to one, the slender body approximation is violated; consequently, the problem should be reinvestigated using the exact Newtonian expression for the pressure coefficient, that is, the sine square law.

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TABLE 1

BOUNDARY CONDITIONS

Quantities given	C_1	C_2	C_3	C	$\lambda_{4f} + 3\dot{y}_f^2$	Additional relationships
d	-1	0	0	0		$\dot{y}_f = \sqrt[3]{C_f/4}$
A	-1		0	0	0	$\lambda_{4f} = 0$, $\dot{y}_f = 0$, $C_f = C_2 d$
M	-1	0		0	0	$\lambda_{4f} = 0$, $\dot{y}_f = 0$, $C_f = C_3 d^2/2$
d, A	-1		0	0		$\dot{y}_f = \sqrt[3]{\frac{C_f - C_2 d}{4}}$
d, M	-1	0		0		$\dot{y}_f = \sqrt[3]{\frac{2C_f - C_3 d^2}{8}}$
A, M	-1			0	0	$\lambda_{4f} = 0$, $\dot{y}_f = 0$, $C_f = C_2 d + C_3 d^2/2$

TABLE 2

NONDIMENSIONAL BOUNDARY CONDITIONS

Quantities given	K_2	K_3	σ_f	$\frac{\tau\dot{\eta}_f}{\sqrt[3]{2C_f}}$	Number of independent parameters
d	0	0		1	0
A	1	0	0	0	0
M	0	1	0	0	0
d, A		0		$\sqrt[3]{1-K_2}$	1
d, M	0			$\sqrt[3]{1-K_3}$	1
A, M	$1-K_3$		0	0	1

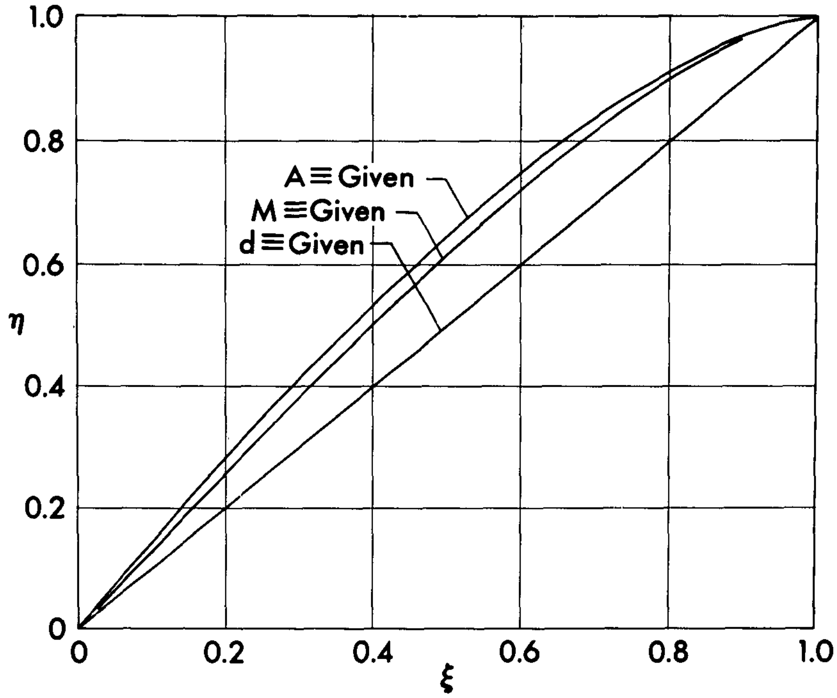

Fig. 1. Optimum shapes for given values of the thickness, the enclosed area, or the moment of inertia of the contour.

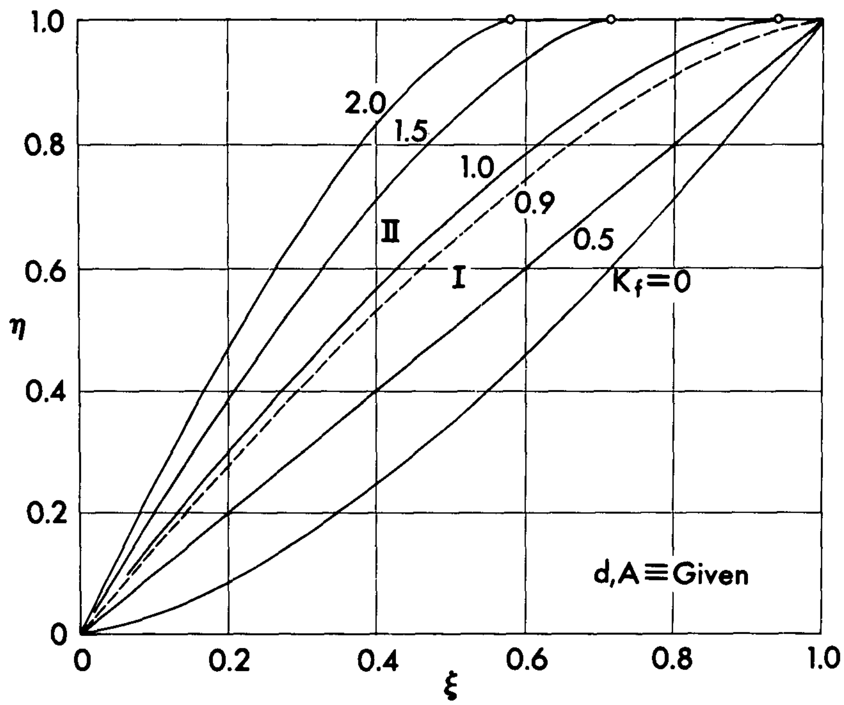

Fig. 2. Optimum shapes for given thickness and enclosed area.

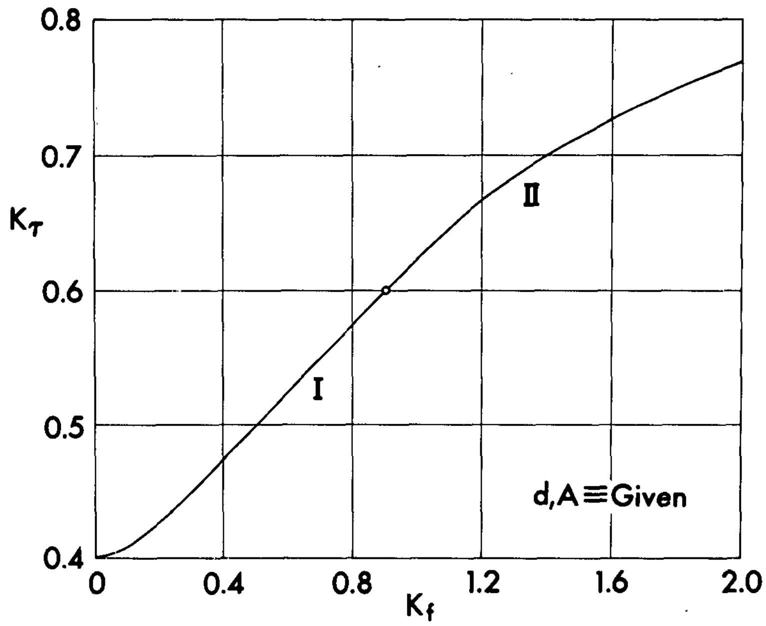

Fig. 3. Thickness ratio for given thickness and enclosed area.

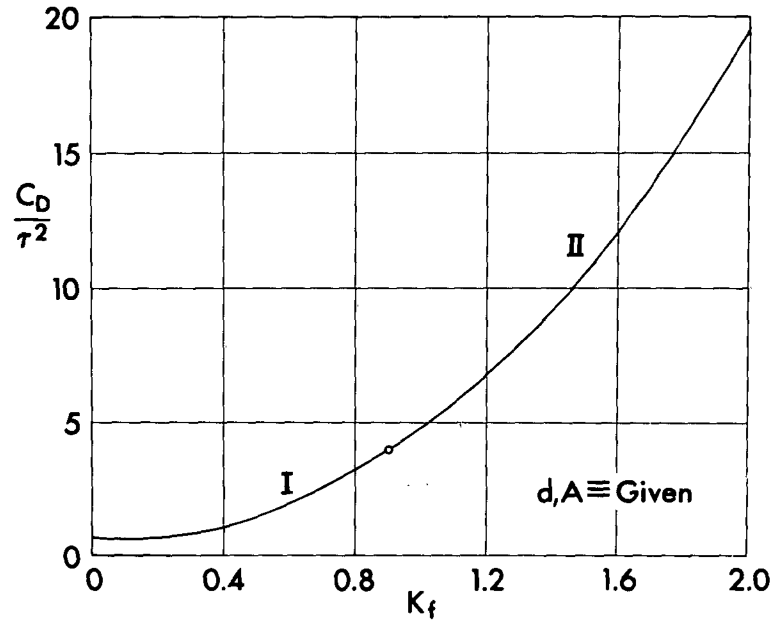

Fig. 4. Drag coefficient for given thickness and enclosed area.

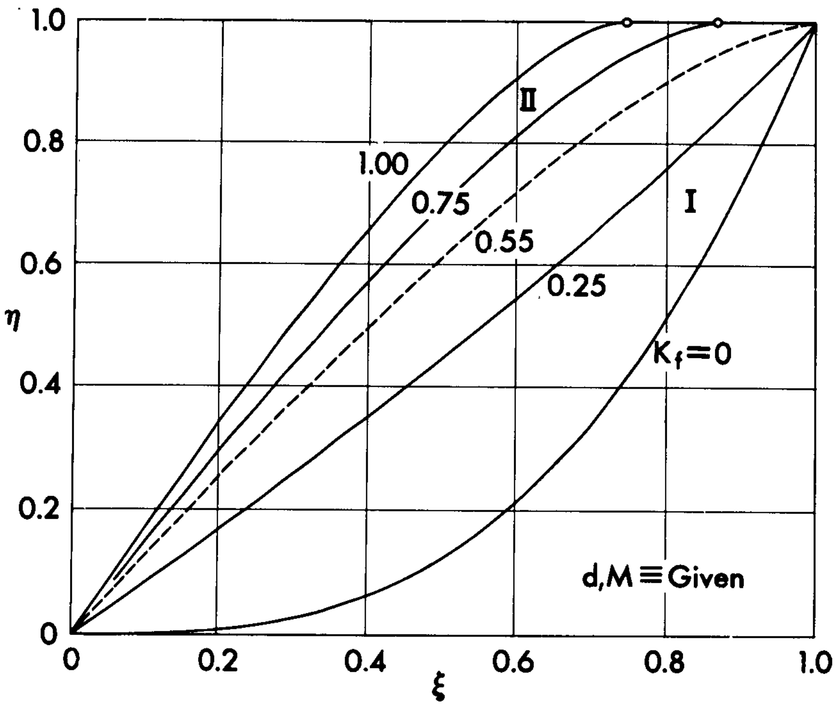

Fig. 5. Optimum shapes for given thickness and moment of inertia of the contour.

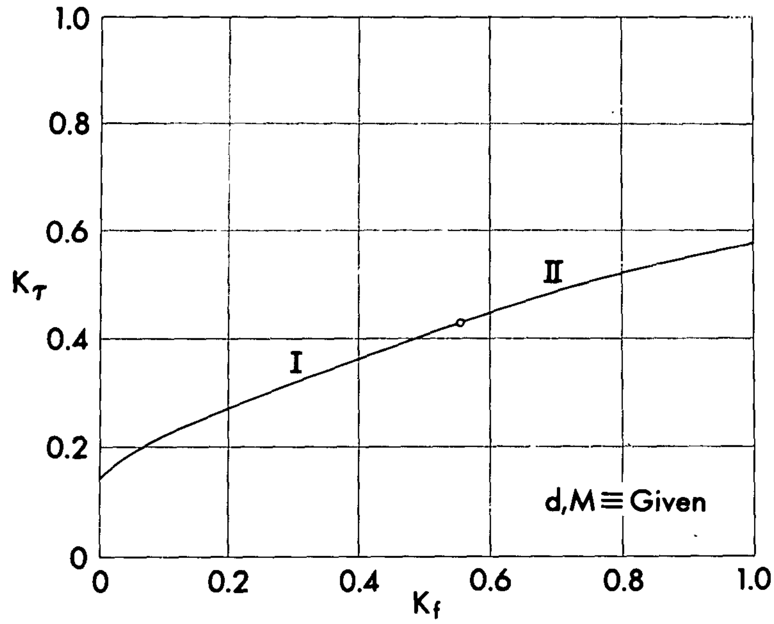

Fig. 6. Thickness ratio for given thickness and moment of inertia of the contour.

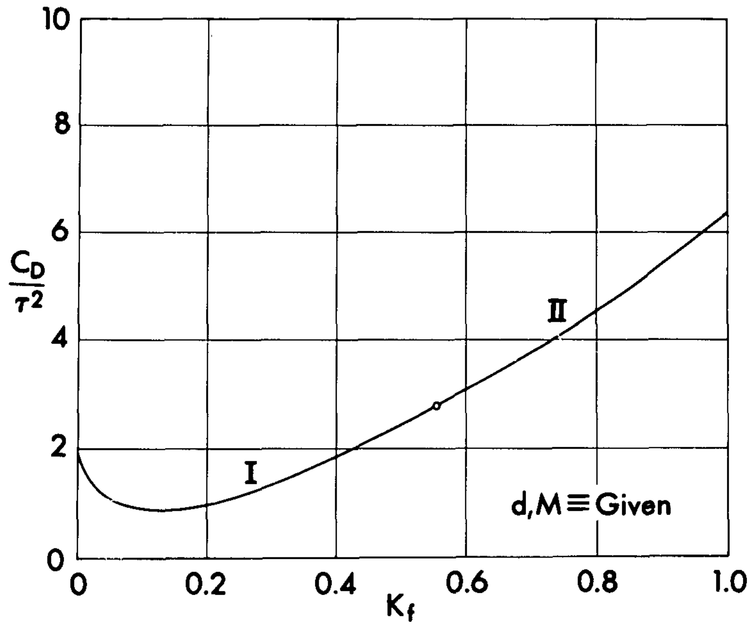

Fig. 7. Drag coefficient for given thickness and moment of inertia of the contour.

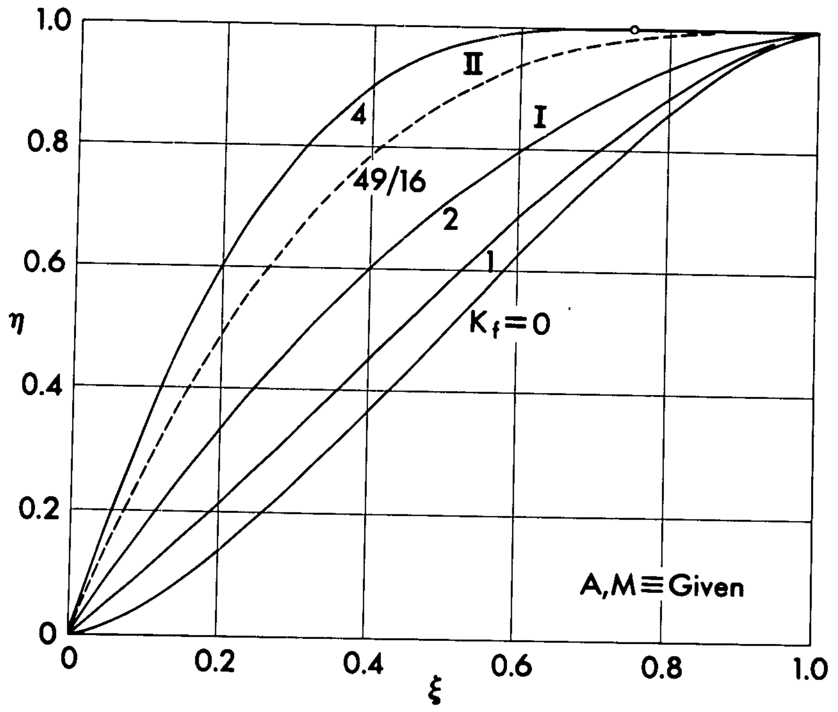

Fig. 8. Optimum shapes for given enclosed area and moment of inertia of the contour.

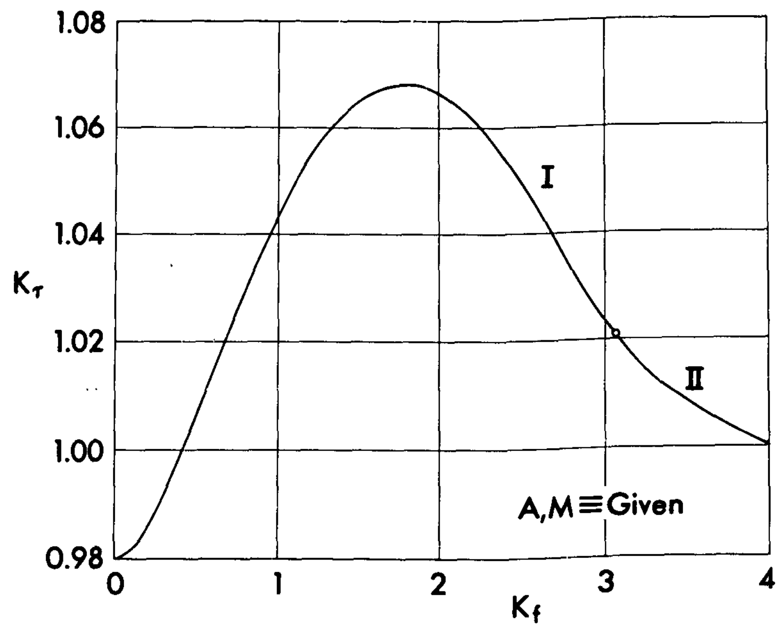

Fig. 9. Thickness ratio for given enclosed area and moment of inertia of the contour.

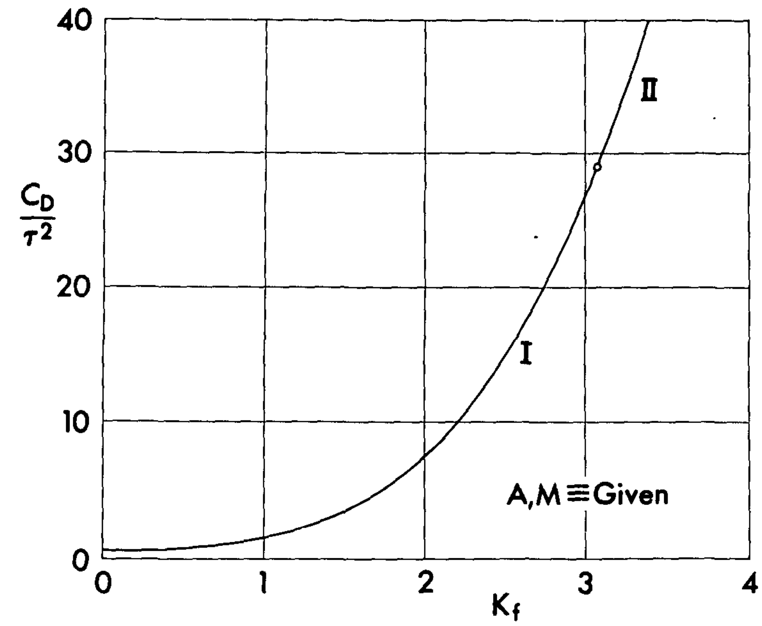

Fig. 10. Drag coefficient for given enclosed area and moment of inertia of the contour.